

SIGNED QUIVERS, SYMMETRIC QUIVERS, AND ROOT SYSTEMS.

D.A.SHMELKIN

ABSTRACT. We define a special sort of weighted oriented graphs, signed quivers. Each of these yields a symmetric quiver, i.e., a quiver endowed with an involutive anti-automorphism and the inherited signs. We develop a representation theory of symmetric quivers, in particular we describe the indecomposable symmetric representations. Their dimensions constitute root systems corresponding to certain symmetrizable generalized Cartan matrices.

1. INTRODUCTION.

Let Q be a finite quiver, i.e., an oriented graph. Fix the notation as follows. We denote by Q_0 and Q_1 the sets of the vertices and the arrows of Q , respectively. For any arrow $\varphi \in Q_1$ denote by $t\varphi$ and $h\varphi$ its tail and its head, respectively. A representation V of Q over \mathbf{k} consists in defining a vector space $V(i)$ over \mathbf{k} , for any $i \in Q_0$, and a \mathbf{k} -linear map $V(\varphi) : V(t\varphi) \rightarrow V(h\varphi)$, for any $\varphi \in Q_1$. The dimension vector $\dim V$ is the collection of $\dim V(i), i \in Q_0$. For a fixed dimension α , we may fix $V(i) = \mathbf{k}^{\alpha_i}$. Then the set $R(Q, \alpha)$ of the representations of dimension α is converted into the vector space

$$(1) \quad R(Q, \alpha) = \bigoplus_{\varphi \in Q_1} \text{Hom}(\mathbf{k}^{\alpha_{t\varphi}}, \mathbf{k}^{\alpha_{h\varphi}}).$$

A homomorphism H of a representation U of Q to another representation, V is a collection of linear maps $H(i), U(i) \rightarrow V(i) \in Q_0$ such that for any $\varphi \in Q_1$ holds $V(\varphi)H(t\varphi) = H(h\varphi)U(\varphi)$. The endomorphisms, automorphisms, and isomorphisms are defined naturally. An easy but very fruitful observation is that the isomorphism classes of representations of Q are the orbits of a reductive group

$$(2) \quad GL(\alpha) = \prod_{i \in Q_0} GL(\alpha_i)$$

acting naturally on $R(Q, \alpha)$, as follows

$$(3) \quad (g(V))(\varphi) = g(h\varphi)V(\varphi)(g(t\varphi))^{-1}.$$

Futhermore, by the Krull-Schmidt theorem, each representation has a unique, modulo isomorphisms and permutations of summands, decomposition into *indecomposable* ones. The classification of representations modulo isomorphism is therefore reduced to that for the indecomposable ones. The latter problem is solved for the *finite* and *tame* quivers. Moreover, by Kac's Theorem [Kac2] the dimensions of the indecomposable representations are exactly the positive roots of the symmetric Kac-Moody algebra corresponding to the underlying graph of Q .

From the point of view of Invariant Theory, the above notation introduces a nice set of reductive linear groups, $(GL(\alpha), R(Q, \alpha))$. This set is nice because the developed language of quiver representations allows (at least in some cases) to describe orbits, invariants, semi-invariants etc. Another important feature of this set is that the underlying quiver Q determines many natural properties of the groups, for example, if Q is finite, then $(GL(\alpha), R(Q, \alpha))$ contains finitely many orbits for any α .

The above set of groups was extended in [DW] to that of *generalized quivers*. These can be described as actions of certain reductive groups in the spaces of either orthogonal or symplectic representations of a *symmetric* quiver S , i.e. a quiver with an involutive anti-automorphism. In this setting one can also classify the orbits in terms of the indecomposable representations; moreover, an important result of [DW, Proposition 2.7] states that each indecomposable orthogonal or symplectic representation is either indecomposable as a representation of S , or is a sum of two indecomposable representations in involution.

We suggest another extension of the set of groups containing all the generalized quivers. Similarly to the usual quiver context, these groups are parametrized by a *signed quiver* and a dimension vector. We represent these groups as actions on the spaces of *symmetric* representations of a symmetric (in the same sense as in [DW]) quiver endowed with signs. A key observation 3.15 is that symmetric representations are conjugate by our group if and only if they are isomorphic as quiver representations. We obtain an a priori description 4.5 of the indecomposable symmetric representations in the same style as in [DW]. As a corollary we get in 4.7 the uniqueness of the decomposition for the symmetric representations.

Analogously to the usual quiver context, the dimensions of indecomposable symmetric representations turn out to be related to the root system for a *symmetrizable* generalized Cartan matrix. Let Γ be the underlying graph of our symmetric quiver and assume that the corresponding involutive automorphism π is *admissible* (see Definition 6.1). Then π acts on the root system $\Delta(\Gamma)$ and one may consider the set consisting of the π -invariant roots and of the sums $\alpha + \pi(\alpha)$ for non-invariant roots α . Recently it was observed in [Hu] that this set is the root system of symmetrizable type corresponding to a graph, which is a sort of factorization of Γ by π . In fact this observation goes back to Kac: we can not find it as a statement but it is used in [Kac3, 7.9] in order to construct the root systems for B_l, C_l, F_4, G_2 in terms of those for $D_{l+1}, A_{2l-1}, E_6, D_4$, respectively. The statement from [Hu] concerns not all signed quivers because it does not depend on the signs. We then generalize this: we define a graph $\Gamma_{\pi, \sigma}$ depending on both π and the signs σ and obtain in 6.8 a description of the root system $\Delta(\Gamma_{\pi, \sigma})$ in terms of $\Delta(\Gamma)$, similar to the above one. In 6.10 we give a sufficient condition for the set of dimensions of indecomposable symmetric representations to be equal to $\Delta(\Gamma_{\pi, \sigma})$.

As for quivers and generalized quivers, we define finite and tame signed quivers and classify them (7.2, 8.2). We describe the indecomposable representations of finite and tame quivers. By 5.9, for any signed quiver this description can be reduced to the same for a quiver such that π is admissible. Considering the finite and tame quivers case by case we construct the indecomposable symmetric representations explicitly and check that the condition of 6.10 is fulfilled. Thus we get in 8.11 that five tame signed quivers yield the root systems corresponding to the graphs: $D_n^{(2)}, C_n^{(1)}, A_{2n}^{(2)}, B_n^{(1)}, A_{2n-1}^{(2)}$.

2. SIGNED QUIVERS.

Definition 2.1. A signed quiver is a triple $Q^\sigma = (Q, \sigma, m)$, where Q is a quiver, $\sigma : Q_0 \cup Q_1 \rightarrow \{-1; 0; 1\}$ is a sign function, $Q_0 = \{1, 1^*, \dots, m, m^*, m+1, \dots, n\}$ contains $m \geq 0$ pairs of vertices that we call *twins*. We also call a vertex i *signed* if $\sigma(i) \neq 0$; otherwise we call it unsigned. The same terms apply to the arrows. The data is subject to the axioms:

1. twins are unsigned
2. a non-loop arrow is signed if and only if its vertices are twins
3. a loop is signed if and only if its vertex is signed.

To draw a signed quiver, we draw the underlying quiver and provide the signed vertices and arrows with their signs; vertices connected by signed arrows are assumed to be twins. For a pair of twins that are not connected, we have to index them explicitly as i and i^* for some i .

Definition 2.2. We say that a dimension $\alpha \in \mathbf{Z}_+^{Q_0}$ is sign-matched, if $\alpha_i = \alpha_{i^*}$ for twins i and i^* and α_j is even, if $\sigma(j) = -1$.

Definition 2.3. Let Q^σ be a signed quiver, let α be a sign-matched dimension. Fix the data as follows:

- a vector space $V(i)$ for each $i \in Q_0$ such that for twins i, i^* , $V(i^*) = V(i)^*$
- for any signed $k \in Q_0$, a linear map $J_k : V(k) \rightarrow V(k)^*$ such that $J_k^* = \sigma(k)J_k$.

Denote by $G(Q^\sigma, \alpha)$ the direct product of groups as follows:

- $GL(V(k))$, for every single unsigned vertex k and every pair k, k^* of twins,
- $O(V(i))$, for every $i \in Q_0$ with $\sigma(i) = 1$,
- $Sp(V(j))$, for every $j \in Q_0$ with $\sigma(j) = -1$.

Denote by $R(Q^\sigma, \alpha)$ the direct sum of $G(Q^\sigma, \alpha)$ -modules as follows:

- $\text{Hom}(V(i), V(j))$, for an unsigned arrow $i \rightarrow j$,
- $S^2V(i)^*$, for $i \xrightarrow{+} i^*$ or a $+$ loop on signed i ; $S^2V(i)$, for $i \xleftarrow{+} i^*$,
- $\wedge^2V(i)^*$, for $i \xrightarrow{-} i^*$ or a $-$ loop on signed i ; $\wedge^2V(i)$, for $i \xleftarrow{-} i^*$.

The group $(G(Q^\sigma, \alpha), R(Q^\sigma, \alpha))$ is in general neither connected nor semi-simple. Denote by $(S(Q^\sigma, \alpha), R(Q^\sigma, \alpha))$ the commutant of its unity connected component (this is a result of replacing the factors $GL(V(i))$ and $O(V(j))$ of $G(Q^\sigma, \alpha)$ by $SL(V(i))$ and $SO(V(j))$, respectively). It is not difficult to show that the set of connected semi-simple groups that we obtain this way can be defined differently, as follows:

Proposition 2.4. *Let (H, U) be a connected semi-simple linear group such that each irreducible H -factor of U is isomorphic (as a linear group) to one of the items:*

$$\begin{aligned}
 & (SL(L) \times SL(M), \text{Hom}(L, M)), (SL(L) \times SO(M), \text{Hom}(L, M)), \\
 & (SL(L) \times Sp(M), \text{Hom}(L, M)), (SL(L), \wedge^2 L), (SL(L), S^2 L), \\
 (4) \quad & (SO(L) \times SO(M), \text{Hom}(L, M)), (SO(L) \times Sp(M), \text{Hom}(L, M)), \\
 & (Sp(L) \times Sp(M), \text{Hom}(L, M)), (SO(L), \wedge^2 L), (Sp(L), S^2 L) \\
 & (SL(L), \text{End}_0(L)), (Sp(L), \wedge_0^2 L), (SO(L), S_0^2 L).
 \end{aligned}$$

Then there exist a signed quiver Q^σ and a sign-matched dimension α such that $(H, U/U^H) \cong (S(Q^\sigma, \alpha), R(Q^\sigma, \alpha)/R(Q^\sigma, \alpha)^{S(Q^\sigma, \alpha)})$.

Remark 2.1. The signed quivers generalize the *generalized quivers* from [DW]. Namely, the generalized quivers are the groups $(G(Q^\sigma, \alpha), R(Q^\sigma, \alpha))$ for the signed quivers of the special sort:

- orthogonal case: all signed arrows are - and all signed vertices are +;
- symplectic case: all signed arrows are + and all signed vertices are -.

Remark 2.2. One of the natural tasks of Invariant Theory consists in classifying actions of reductive groups with nice properties of orbits and invariants. Examining the known classifications, one can see that an important part of the lists consists of the groups arising from signed quivers, especially in what concerns "serial" cases. For example the connected irreducible linear groups with finitely many orbits are classified in [Kac1], Theorem 2; one can observe that all infinite series of groups arise from signed quivers except for the unique example:

$$(SL_2 \times SL_3 \times SL_n, \mathbf{k}^2 \otimes \mathbf{k}^3 \otimes \mathbf{k}^n), n \geq 3.$$

Analogously, almost all serial irreducible semi-simple linear groups with polynomial algebra of invariants (see [Lit]) arise from signed quivers.

3. SIGNED QUIVERS AND SYMMETRIC QUIVERS.

The subsequent definition follows that of Derksen and Weyman:

Definition 3.1. (cf. [DW]) A symmetric quiver S is a quiver endowed with an involution $*$ acting on both S_0 and S_1 such that

$$t(\varphi^*) = (h\varphi)^*, h(\varphi^*) = (t\varphi)^*.$$

For a signed quiver Q^σ , we first define $*$ on subsets of Q_0 and Q_1 : $i^* = i^*$, $(i^*)^* = i$ for a pair i, i^* of twins, $k^* = k$ for a signed vertex k ; $\varphi^* = \varphi$, for a signed arrow φ . To define $*$ on the whole of Q we have to extend Q , as follows: we add a vertex j^* for any single unsigned vertex j of Q and an arrow $\varphi^* : q^* \rightarrow p^*$ for any unsigned arrow $\varphi : p \rightarrow q$ of Q^σ . We denote by \tilde{Q} the quiver that we obtained this way. Clearly, the defined involution $*$ fulfills definition 3.1.

Example 3.2. Let Q^σ be as follows: $\circ \rightarrow \circ \leftarrow \circ \xrightarrow{-} \circ$. We numerate the vertices from left to right as 1, 2, 3, 3*. Then \tilde{Q} has vertices 1, 2, 3, 3*, 2*, 1* and looks like: $\circ \rightarrow \circ \leftarrow \circ \xrightarrow{-} \circ \leftarrow \circ \rightarrow \circ$.

We therefore constructed a symmetric quiver \tilde{Q} , which is also signed such that all signed vertices and arrows belong to Q^σ . The key tool in dealing with the representations of \tilde{Q} is the duality functor. For $\varphi \in \tilde{Q}_1$ set $s_\varphi = -1$ if either $\sigma(\varphi) = -1$ and $\sigma(h\varphi) \neq -1$ or $\sigma(\varphi) \neq -1$ and $\sigma(h\varphi) = -1$, and $s_\varphi = 1$, otherwise.

Definition 3.3. Let V be a representation of \tilde{Q} . The dual representation V^* is:

$$V^*(i) = (V(i^*))^*, i \in S_0,$$

$$V^*(\varphi) = s_\varphi V(\varphi^*)^* : (V(h\varphi^*))^* \rightarrow V(t\varphi^*)^*, \varphi \in S_1.$$

Remark 3.1. The sign $*$ is used in the above definition in three different senses: for the involution of \tilde{Q} , for the dual vector space, and for the dual linear map.

Remark 3.2. Our definition of the dual representation differs from that from [DW] by the multiple s_φ . This difference allows us to recover a wider set of groups.

Remark 3.3. Clearly, we have: $(V^*)^*(\varphi) = s_\varphi s_{\varphi^*} V(\varphi)$, and $s_\varphi s_{\varphi^*} = -1$ when exactly one of $t\varphi, h\varphi$ is a $-$ vertex. So $(V^*)^*$ is not always the same as V . However $(V^*)^*$ is always conjugate to V by the product of $-\text{Id}$ operators taken over all $-$ vertices of Q^σ .

Definition 3.4. We say that a dimension $\gamma \in \mathbf{Z}_+^{\tilde{Q}_0}$ is sign-matched, if $\gamma_i = \gamma_{i^*}$ and γ_j is even, if $\sigma(j) = -1$.

Clearly, extending by symmetricity a sign-matched dimension α for Q to a dimension vector $\tilde{\alpha} \in \mathbf{Z}_+^{\tilde{Q}_0}$ we get $\tilde{\alpha}$ is sign-matched. We define an embedding of the space $R(Q^\sigma, \alpha)$ to $R(\tilde{Q}, \tilde{\alpha})$. Recall that the definition of $R(Q^\sigma, \alpha)$ is based, in particular on fixing vector spaces $V(i), i \in Q_0$ such that $V(i^*) = V(i)^*$ for a pair i, i^* of twins, and on a choice of the isomorphism $J_k : V(k) \rightarrow V(k)^*$ for any signed vertex k . For any vertex $i \in \tilde{Q}_0$ set $V(i) = \mathbf{k}^{\tilde{\alpha}_i}$ and choose a structure as follows:

Definition 3.5. A *signed form* on the spaces $V(i), i \in \tilde{Q}_0$ is a collection of isomorphisms $J_i : V(i) \rightarrow (V(i^*))^*, i \in \tilde{Q}_0$, such that $J_{i^*} = (J_i)^*$, if $i \neq i^*$, and $(J_i)^* = \sigma(i)J_i$, otherwise.

Proposition 3.6. All signed forms are conjugate by $GL(\tilde{\alpha})$.

Proof. This follows from the fact that all non-degenerate symmetric and anti-symmetric forms on a vector space W are conjugate by $GL(W)$ and (since J_{i^*} is determined by J_i) that all non-degenerate maps $U \rightarrow W^*, \dim U = \dim W$ are conjugate by $GL(U) \times GL(W)$. \square

So all signed forms are equivalent to each other and once a form is chosen, we are able to introduce our main concept:

Definition 3.7. A representation $V \in R(\tilde{Q}, \tilde{\alpha})$ is called symmetric, if the collection of maps $\{J_i : V(i) \rightarrow V^*(i), i \in \tilde{Q}_0\}$ is an isomorphism of V to V^* .

Example 3.8. Let Q^σ (and also \tilde{Q}) consist of one $+$ vertex a and one $-$ loop φ on it. By Definition 3.3, we have for a representation V : $V^*(\varphi) = -V(\varphi)^*$. A signed form J_a is in this case just a symmetric form on $V(a)$. The symmetric representations are subject to the equation: $V(\varphi)^* J_a + J_a V(\varphi) = 0$ or for $A = J_a V(\varphi) : A^* = -A$, hence, these are the anti-self-adjoint endomorphisms of $V(a)$. Note that the set of symmetric representations is a $O(V(a))$ -stable vector subspace in $\text{End}(V(a))$ isomorphic to $\wedge^2 V(a)$.

Example 3.9. Let Q^σ (and also \tilde{Q}) consist of one $+$ vertex a and one $+$ loop φ on it. The symmetric representations are the self-adjoint endomorphisms and constitute a $O(V(a))$ -module $S^2 V(a)$.

Example 3.10. Let Q^σ (and also \tilde{Q}) be $\circ \xrightarrow{-} \circ$. Denote by φ the unique arrow, $a = t\varphi, b = h\varphi$. By Definition 3.3, we have for a representation V : $V^*(\varphi) = -V(\varphi)^*$. The symmetric representations are subject to the equation: $V(\varphi)^* J_a + J_b V(\varphi) = 0$ or for $A = J_b V(\varphi) : A^* = -A$. So the set of symmetric representations is naturally isomorphic to $\wedge^2 V(a)^*$.

We define an involution $\tau \in GL(R(\tilde{Q}, \tilde{\alpha}))$; namely, for any arrow $\varphi \in \tilde{Q}_1$ set

$$(5) \quad \tau(V)(\varphi) = J_{h\varphi}^{-1} V^*(\varphi) J_{t\varphi} \in \text{Hom}(V(t\varphi), V(h\varphi)).$$

Clearly, a representation $V \in R(\tilde{Q}, \tilde{\alpha})$ is symmetric if and only if $\tau(V) = V$.

Proposition 3.11. $\tau^2 = \text{Id}$.

Proof. $\tau^2(V)(\varphi) = J_{h\varphi}^{-1}((\tau(V))^*(\varphi)J_{t\varphi}) = J_{h\varphi}^{-1}s_\varphi(\tau(V)(\varphi^*))^*J_{t\varphi} =$

$$J_{h\varphi}^{-1}s_\varphi(J_{h\varphi}^{-1}s_{\varphi^*}(V(\varphi))^*J_{t\varphi^*})^*J_{t\varphi} = s_\varphi s_{\varphi^*}J_{h\varphi}^{-1}J_{(h\varphi)^*}^*V(\varphi)J_{(t\varphi)^*}^*J_{t\varphi},$$

(here and below we write X^{-*} instead of $(X^*)^{-1} = (X^{-1})^*$). Note that $J_{h\varphi}^{-1}J_{(h\varphi)^*}^*$ (resp. $J_{(t\varphi)^*}^*J_{t\varphi}$) is the ± 1 scalar operator, -1 iff $\sigma(h\varphi) = -1$ (resp. $\sigma(t\varphi) = -1$). By Remark 3.3, $s_\varphi s_{\varphi^*} = -1$ when exactly one of $t\varphi$ and $h\varphi$ is a $-$ vertex. \square

Proposition 3.12. For $g \in GL(\tilde{\alpha})$, $g^\tau = \tau g \tau$ belongs to the image of $GL(\tilde{\alpha})$ in $GL(R(\tilde{Q}, \tilde{\alpha}))$, $g_i^\tau = J_i^{-1}g_{i^*}^{-*}J_i$.

Proof. First consider an unsigned arrow $\varphi \in Q_1$, $t\varphi = k$, $h\varphi = l$. Set $A = V(\varphi)$, $B = V(\varphi^*)$. Then we have:

$$\tau(A, B) = (s_\varphi J_l^{-1}B^*J_k, s_{\varphi^*}J_k^{-1}A^*J_l^*), g(A, B) = (g_l A g_k^{-1}, g_k^* B g_l^{-*}).$$

$$\tau g \tau(A, B) = (s_\varphi s_{\varphi^*}J_l^{-1}g_l^{-*}J_l^* A J_k^{-*}g_k^*J_k, s_{\varphi^*}s_\varphi J_k^{-1}g_k^{-*}J_k^* B J_l^{-*}g_l^*J_l^*),$$

and we are done by Remark 3.3. Similarly, we consider a signed φ . \square

Corollary 3.13. The centralizer $GL(\tilde{\alpha})^\tau$ of τ is generated by the kernel for the action of $GL(\tilde{\alpha})$ and the subgroup

$$(6) \quad G^\tau = \{g \in GL(\tilde{\alpha}) \mid J_i^{-1}g_{i^*}^{-*}J_i = g_i, i \in \tilde{Q}_0\}.$$

Corollary 3.14. The linear groups $(GL(\tilde{\alpha})^\tau, R(\tilde{Q}, \tilde{\alpha})^\tau)$ and $(G(Q^\sigma, \alpha), R(Q^\sigma, \alpha))$ are isomorphic.

Proof. By Corollary 3.13, the groups $(GL(\tilde{\alpha})^\tau, R(\tilde{Q}, \tilde{\alpha})^\tau)$ and $(G^\tau, R(\tilde{Q}, \tilde{\alpha})^\tau)$ are equal. Clearly, G^τ is naturally isomorphic to $G(Q^\sigma, \alpha)$, so we only need to compare Definition 2.3 of $R(Q^\sigma, \alpha)$ and Definition 3.7. By Definition 3.7, the map $V(\varphi^*)$ is uniquely defined by $V(\varphi)$ for any symmetric representation V and any unsigned arrow $\varphi \in Q_1$. Since this is the unique restriction for $V(\varphi)$ and $V(\varphi^*)$, we get the submodule $\text{Hom}(V(t\varphi), V(h\varphi)) \subseteq R(\tilde{Q}, \tilde{\alpha})^\tau$. If φ is signed, then in Examples 3.8, 3.9, 3.10 we saw that the symmetricity condition of $V(\varphi)$ gives rise to a submodule as in Definition 2.3. \square

Theorem 3.15. Symmetric representations are conjugate by G^τ if and only if they are isomorphic.

Proof. The proof is based on an observation from [MWZ], as follows. Let X be a set acted upon by a group G and an involution σ , which normalizes G . Assume furthermore that G is a subgroup in the group of invertible elements of a finite-dimensional associative algebra A and the anti-automorphism $g \rightarrow \sigma g^{-1} \sigma$ of G extends to a \mathbf{k} -linear involution of A .

Proposition 3.16. [MWZ] Suppose that for any $x \in X$ each invertible element of the linear span in A of the stabilizer G_x belongs to G_x . Then any two points in X^σ are G -conjugate if and only if they are $Z_G(\sigma)$ -conjugate.

We apply Proposition 3.16 to $X = R(\tilde{Q}, \tilde{\alpha})$, $A = L(\tilde{\alpha})$, $G = GL(\tilde{\alpha})$ and $\sigma = \tau$. Actually, by Proposition 3.12, $(\tau g^{-1} \tau)_i = J_i^{-1}g_{i^*}^{-*}J_i$ hence, this anti-automorphism of $GL(\tilde{\alpha})$ extends to the whole of $L(\tilde{\alpha})$. For any $V \in X$, $G_V = \text{Aut}(V)$ and its linear span is $\text{End}(V)$. It remains to note that by Corollary 3.13, the action of G^τ on $R(\tilde{Q}, \tilde{\alpha})$ is equal to that of $Z_G(\sigma) = GL(\tilde{\alpha})^\tau$. \square

Definition 3.17. Let U, V be symmetric representations of \tilde{Q} with respect to signed form J^U, J^V , respectively. We say that U and V are *symmetrically isomorphic* if there exists an isomorphism $T : U \rightarrow V$ such that T is an isometry of the signed forms: for any $x \in U(i), y \in U(i^*)$ holds: $\langle J_i^U(x), y \rangle = \langle J_i^V(Tx), Ty \rangle$.

Proposition 3.6 and Theorem 3.15 imply:

Corollary 3.18. *Symmetric representations are symmetrically isomorphic if and only if they are isomorphic.*

4. INDECOMPOSABLE SYMMETRIC REPRESENTATIONS.

In this section we introduce and describe the indecomposable symmetric representations of the symmetric quiver \tilde{Q} corresponding to a signed quiver Q^σ . Again these results and their proofs generalize those from [DW].

Let V^i be a symmetric representation of \tilde{Q} with respect to a signed form J^i , $i = 1, 2$. Then $V_1 + V_2$ is symmetric with respect to the signed form $J = J^1 \oplus J^2$.

Definition 4.1. A symmetric representation V of \tilde{Q} is called *indecomposable* if V is not isomorphic to a non-trivial direct sum $W = U + V$ of symmetric representations.

By Corollary 3.18, a symmetric representation is determined, modulo symmetric isomorphism by the isomorphism classes of indecomposable symmetric summands of any splitting. It is however not clear that all splittings of a representation into indecomposable ones have the same summands modulo permutations and isomorphisms, as it is for the usual quivers. It follows directly from Definition 3.7 of symmetric representations:

Definition-Proposition 4.2. Let V be a symmetric representation of \tilde{Q} , $W \subseteq V$ be a subrepresentation. Set $W_i^\perp = J_i^{-1}(W_{i^*}^\perp) \subseteq V_i$, where $W_{i^*}^\perp \subseteq V_{i^*}^*$ is the annihilator. This collection of subspaces is a subrepresentation in V .

Proposition 4.3. *Let V be a symmetric representation of \tilde{Q} , $W \subseteq V$ be a subrepresentation. If $W_i \cap W_i^\perp = 0$ for any $i \in \tilde{Q}_0$, then both W and W^\perp carry structures of symmetric representations such that V splits: $V = W + W^\perp$.*

Proof. Assuming $W_i \cap W_i^\perp = 0$, we get an inequality $\dim W_i + \dim W_i^\perp \leq \dim V_i$, which can be rewritten as $\dim W_i \leq \dim W_{i^*}$ (because $\dim V_i = \dim V_{i^*}$). These inequalities for i and i^* yield: $\dim W_i = \dim W_{i^*}$. Hence, $\dim W_i + \dim W_i^\perp = \dim V_i$ and $V = W + W^\perp$ splits as a representation of \tilde{Q} . Furthermore, the condition $W_i \cap W_i^\perp = 0$ implies $J_i W_i$ is naturally isomorphic to $W_{i^*}^*$. Hence, restricting the form J_i to W_i for all i , we get a signed form on W . Similarly, we endow W^\perp with a structure of symmetric representation. \square

There is a natural way to construct symmetric representations, as follows:

Proposition 4.4. *For any representation V of \tilde{Q} , $V + V^*$ carries a natural structure of symmetric representation.*

Proof. For $X = V + V^*$, $X(i) = V(i) \oplus V(i^*)^*$. Hence, $(X(i^*))^* = X(i)$ and we set $J_i = \text{Id}$ for $i \neq i^*$; clearly $J_i^* = J_{i^*}$. If i is signed, then we define

$$J_i : V(i) \oplus V(i)^* \rightarrow V(i)^* \oplus V(i), J_i(v + v^*) = \sigma(i)v^* + v$$

for any $v \in V(i), v^* \in V(i)^*$. In this case we have: $J_i^*(v + v^*) = v^* + \sigma(i)v = \sigma(i)J_i$. A straightforward calculation shows that J is an isomorphism of X to X^* . \square

Lemma 4.5. *Let V be an indecomposable symmetric representation of \tilde{Q} . Then as a representation of \tilde{Q} V is either indecomposable or $V = W + W^*$ for an indecomposable W .*

Proof. Assume that V splits as a representation of \tilde{Q} : $V = W + U$ such that W is indecomposable. By Proposition 4.3, $W_i \cap W_i^\perp \neq 0$ for some i .

Note that we have another splitting: $V = W^\perp + U^\perp$. First we claim: $W_i \cap U_i^\perp = 0$ for any i . Consider two maps:

$$(7) \quad P_1 : U^\perp \hookrightarrow V = W + U \xrightarrow{P_1} W, \quad P_2 : W \hookrightarrow V = W^\perp + U^\perp \xrightarrow{P_2} U^\perp$$

Clearly, P_1 and P_2 are homomorphisms of representations of \tilde{Q} , hence, $P_1 P_2$ is an endomorphism of W . The kernel of P_2 is equal to $W \cap W^\perp$, hence, is nonempty. So $P_1 P_2$ is not invertible, hence, should be nilpotent, because the endomorphism ring of the indecomposable representation W is local. On the other hand, any element of $W_i \cap U_i^\perp$ is stable under both P_1 and P_2 . Thus $W_i \cap U_i^\perp = 0$ for any i .

Note that, as representation of \tilde{Q} , $U^\perp \cong (V/U)^* \cong W^*$. Then the dimension of $X = W + U^\perp$ is symmetric. We claim: $X_i \cap X_i^\perp = 0$ for all i . Consider a map $T : X \rightarrow X$ given by the matrix

$$\begin{pmatrix} \text{Id}_W & P_1 \\ P_2 & \text{Id}_{U^\perp} \end{pmatrix}$$

Taking into account the equality $X_i^\perp = W_i^\perp \cap U_i$, one can easily check that the kernel of T is exactly $X \cap X^\perp$. That $P_1 P_2$ is nilpotent implies the same for $T - \text{Id}_X$, hence, T is invertible and the claim is proved. By Proposition 4.3, V splits as $V = X + Y$, so $V = X = W + U^\perp \cong W + W^*$. \square

Proposition 4.6. *If W is an indecomposable representation of \tilde{Q} such that there exists a symmetric representation $W' \cong W$, then the symmetric representation $W + W^*$ is decomposable.*

Proof. By assumption, $\dim W = \dim W^*$. Hence, we have two symmetric representations of the same dimension: $2W'$ and $W + W^*$. Since these are isomorphic as representations of \tilde{Q} , by Theorem 3.15, they are symmetrically isomorphic. Therefore, $W + W^*$ splits. \square

Theorem 4.7. *For any symmetric representation V of \tilde{Q} there exists a unique, modulo isomorphisms and permutations of factors decomposition of V into a direct sum of indecomposable symmetric representations.*

Proof. By Lemma 4.5, each splitting of V into a sum of indecomposable symmetric representations consists of items W_1 or $W_2 + W_2^*$, where W_1 and W_2 are indecomposable representations of \tilde{Q} . By Proposition 4.6, none W_2 -like summand can be isomorphic to a W_1 -like one. Hence, the summands of all splittings are isomorphic as representations of \tilde{Q} , modulo permutations. By Theorem 3.15, they are also symmetrically isomorphic. \square

5. TWO FUNCTORS.

Definition 5.1. Let Q^σ be a signed quiver and $i \in Q_0$ a signed vertex incident to a unique arrow φ ; denote by j the other vertex of φ and assume $j \neq i$. Replacing i by j^* and φ by a $\sigma(i)$ -signed arrow ψ between j and j^* we get a new signed quiver

that we denote by $\widetilde{\text{Del}}_i(Q)$. For a representation V of \widetilde{Q} denote by $\text{Del}_i(V)$ the representation of $\widetilde{\text{Del}}_i(Q)$ such that $\text{Del}_i(V)(\psi)$ is the composition (in the appropriate order) of $V(\varphi)$ and $V(\varphi^*)$.

Example 5.2. $Q^\sigma : \circ_j \rightarrow \circ_i^+ \quad \widetilde{Q} : \circ_j \rightarrow \circ_i^+ \rightarrow \circ_{j^*} \quad \text{Del}_i(Q) = \widetilde{\text{Del}}_i(Q) : \circ_j \xrightarrow{+} \circ_{j^*}$.

Since the vertices of $\widetilde{\text{Del}}_i(Q)$ are vertices of \widetilde{Q} , a signed form for $\widetilde{\text{Del}}_i(Q)$ can be inherited from \widetilde{Q} . One can easily deduce from definitions:

Proposition 5.3. *If V is a symmetric representation, then $\text{Del}_i(V)$ is.*

Definition 5.4. Let Q^σ be a signed quiver and $\varphi \in Q_1$ a signed non-loop arrow. Inserting a new $\sigma(\varphi)$ -signed vertex a and replacing the arrow φ by an unsigned arrow $\psi : t\varphi \rightarrow a$ we get a new signed quiver that we denote by $\widetilde{\text{Ins}}_\varphi(Q)$. For a representation V of \widetilde{Q} denote by $\text{Ins}_\varphi(V)$ the representation of $\widetilde{\text{Ins}}_\varphi(Q)$ such that $\text{Ins}_\varphi(V)(a) = \text{Im}V(\varphi)$, $\text{Ins}_\varphi(V)(\psi) = V(\varphi)$, and $\text{Ins}_\varphi(V)(\psi^*)$ is the embedding $\text{Im}V(\varphi) \rightarrow V(h\varphi)$.

Example 5.5. $Q^\sigma = \widetilde{Q} : \circ \xrightarrow{\varphi, +} \circ \quad \text{Ins}_\varphi(Q) : \circ \xrightarrow{\psi} \circ_a^+ \quad \widetilde{\text{Ins}}_\varphi(Q) : \circ \xrightarrow{\psi} \circ_a^+ \xrightarrow{\psi^*} \circ$.

Proposition 5.6. *Set $p = \text{Ins}_\varphi(V)(\psi)$, $i = \text{Ins}_\varphi(V)(\psi^*)$. If V is a symmetric representation, then the map $J_a = i^* J_l p^{-1} : V(a) \rightarrow V(a)^*$ is well defined, J_a and the signed form J of \widetilde{Q} yield a signed form for $\widetilde{\text{Ins}}_\varphi(Q)$, and $\text{Ins}_\varphi(V)$ is symmetric.*

Proof. Set $l = t\varphi$, $l^* = h\varphi$, $A = V(\varphi) = ip$. Consider a diagram as follows:

$$\begin{array}{ccccc} V(l) & \xrightarrow{p} & V(a) & \xrightarrow{i} & V(l^*) \\ J_l \downarrow & & \downarrow i^* J_l p^{-1} & & \downarrow J_l^* \\ V(l^*)^* & \xrightarrow{i^*} & V(a)^* & \xrightarrow{\sigma(\varphi)p^*} & V(l)^* \end{array}$$

Since p is surjective by definition, p^{-1} is defined on the whole of $V(a)$, uniquely modulo the kernel of p . Assume that V is symmetric: $\sigma(\varphi)A^*J_l = J_l^*A$. If $p(v) = 0$, then $A(v) = 0$ and $\sigma(\varphi)p^*i^*J_l(v) = \sigma(\varphi)A^*J_l(v) = J_l^*A(v) = 0$. Since p^* is injective, this implies $i^*J_l(v) = 0$, so $J_a = i^*J_l p^{-1}$ is well defined. It is also clear that both squares are commutative. Finally, $J_a^* = p^{-*}J_l^*i$, and since the right square is commutative, $J_l^*i = \sigma(\varphi)p^*J_a$, so $J_a^* = \sigma(\varphi)J_a$. \square

Remark that the functors Del and Ins are inverse to each other for quivers: $\text{Del}_a(\text{Ins}_\varphi(Q)) = Q$, $\text{Ins}_\psi(\text{Del}_i(Q)) = Q$. One can easily prove:

Proposition 5.7. 1. *Assume that the quiver Q is as in 5.4 and V is a symmetric representation of \widetilde{Q} . Then $\text{Del}_a(\text{Ins}_\varphi(V)) = V$.*

2. *Assume that the quiver Q is as in 5.1; set $\gamma = \varphi$ or $\gamma = \varphi^*$ such that $h\gamma = i$. If V is a symmetric representation of \widetilde{Q} such that $V(\gamma)$ is surjective, then $\text{Ins}_\psi(\text{Del}_i(V)) = V$.*

Corollary 5.8. *Assume that the quiver Q is as in 5.4. Then the functor Ins is an isomorphism of the symmetric representations of \widetilde{Q} onto those symmetric representations W of $\widetilde{\text{Ins}}_\varphi(Q)$ such that $W(\psi)$ is surjective.*

Proposition 5.9. *Assume that the quiver Q is as in 5.4. Then the indecomposable symmetric representations of \tilde{Q} are the representations $\text{Del}_a(W)$ for all indecomposable symmetric representations W of $\text{Ins}_\varphi(Q)$ such that $W(\psi)$ is surjective.*

Proof. Clearly both functors Del and Ins respect direct sums of representations. Hence, by Proposition 5.7.2, $\text{Del}_a(W)$ is indecomposable for an indecomposable representation W , and by Proposition 5.7.1, all indecomposable symmetric representations of \tilde{Q} can be obtained this way. \square

6. ROOT SYSTEMS FOR SYMMETRIZABLE GENERALIZED CARTAN MATRICES.

The notion of a generalized Cartan matrix (GCM) and the associated Weyl group and root system was introduced by V.Kac ([Kac2], [Kac3]). In [Kac2] is proved that the root system corresponding to a *symmetric* GCM A is exactly the dimensions of the indecomposable representations of a quiver having the Dynkin diagram $S(A)$ of A as the underlying graph. Moreover, α is a real root if and only if there exists a unique indecomposable of dimension α , modulo isomorphism.

It is of interest to consider not only symmetric GCM but also the *symmetrizable* ones. In particular, all GCM of finite and affine types are symmetrizable (see their classification in [Kac2], [Kac3]). In some cases the root system corresponding to a symmetrizable GCM can be obtained factorizing (in a sense) the root system of a symmetric GCM A by a diagram automorphism of $S(A)$. This was used, in particular, in [Kac3, §7.9] in order to construct the root systems for B_l, C_l, F_4, G_2 in terms of those for $D_{l+1}, A_{2l-1}, E_6, D_4$, respectively. Recently A.Hubery observed in [Hu] that a description of this kind is possible in the context of an *admissible* automorphism of the graph $S(A)$. In particular, for a symmetric quiver \tilde{Q} such that the automorphism $*$ of the underlying graph is admissible (see 6.2) this result gives a root system structure to the set consisting of the symmetric roots and the sums of different roots symmetric to each other. By Lemma 4.5, this set should be related to that of dimensions of the indecomposable symmetric representations of \tilde{Q} , but only in case when all signed vertices are $+$. In this section we suggest a more general approach such that for arbitrary signs we introduce a graph and a kind of symmetrization of roots, both depending on signs, and prove that this symmetrization is the root system corresponding to the graph (Lemma 6.8). Furthermore, we show in Corollary 6.10 that under some extra conditions this root system is precisely the set of dimensions of the indecomposable symmetric representations.

Let Γ be a simply-laced graph and let π be an involutive automorphism of Γ .

Definition 6.1. We say that π is admissible if none edge of Γ is π -stable.

Assume from now on that π is admissible and fix also a sign $\sigma(i) = \pm 1$ for any $i \in \Gamma^\pi$, the set of π -stable vertices of Γ .

Example 6.2. Let Q^σ be a signed quiver that has neither signed arrows nor arrows joining signed vertices. Then the underlying graph Γ of \tilde{Q} together with $\pi = *$ and the sign σ yield a data as above.

Denote by $\Gamma_{\pi, \sigma}$ a graph with vertices corresponding to the π -orbits of the vertices of Γ and the edges as follows (denote by \bar{i} the π -orbit of i):

for an edge of Γ between $i, j \notin \Gamma^\pi$ we draw a simple edge between \bar{i} and \bar{j}

for an edge of Γ between $i \notin \Gamma^\pi$ and $j \in \Gamma^\pi$ we draw a double edge between \bar{i} and \bar{j} , oriented to j (resp. from j) if $\sigma(j) = 1$ (resp. $\sigma(j) = -1$).

Example 6.3. Let Q^σ be $\overset{+}{\circ} \leftarrow \circ \rightarrow \bar{\circ}$. Then Q^σ meets the condition of 6.2 and following the above definition we get a graph $\circ \Leftarrow \circ \Leftarrow \circ$.

We are now going to describe the root system corresponding to the graph $\Gamma_{\pi,\sigma}$ in terms of the root system $\Delta(\Gamma)$. Let $Q(\Gamma) = \mathbf{Z}^{\Gamma_0}$ be the root lattice of Γ and $Q(\Gamma)_+ = \mathbf{Z}_+^{\Gamma_0}$ be the positive vectors. Consider a sublattice as follows:

$$Q(\Gamma)_{\pi,\sigma} = \{\alpha \in Q(\Gamma) \mid \alpha_{\pi(i)} = \alpha_i, \sigma(j)^{\alpha_j} = 1, j \in \Gamma^\pi\}.$$

In other words, we consider the symmetric vectors having even coefficients on $-$ vertices. Note that by Definition 3.4, in the situation of Example 6.2, $Q(\Gamma)_{\pi,\sigma}$ is exactly the set of sign-matched dimension vectors. Denote by $\Pi = \{\varepsilon_i, i = 1, \dots, n\}$ the standard basis of $Q(\Gamma)$ (the simple roots). For $i \notin \Gamma^\pi$ (resp. $\sigma(i) = 1, \sigma(i) = -1$) set $\beta_i = \varepsilon_i + \varepsilon_{\pi(i)}$ (resp. $\varepsilon_i, 2\varepsilon_i$); denote by $\Pi_{\pi,\sigma}$ the obtained set. Clearly, $\Pi_{\pi,\sigma}$ is a basis of $Q(\Gamma)_{\pi,\sigma}$ and also a system of generators for the semi-group $Q(\Gamma)_{\pi,\sigma,+} = Q(\Gamma)_{\pi,\sigma} \cap Q(\Gamma)_+$.

Recall that the Tits form endows the lattice $Q(\Gamma)$ with a bilinear symmetric form $\langle \cdot, \cdot \rangle$ such that its matrix in the basis Π is exactly the GCM matrix of Γ divided by 2. The group $W(\Gamma)$ generated by the simple reflections $r_i(\alpha) = \alpha - 2\langle \alpha, \varepsilon_i \rangle \varepsilon_i$ is called the Weyl group. The elements of $\Delta_+^{re} = W(\Gamma)\Pi \cap Q(\Gamma)_+$ are called *real positive roots*. Furthermore, the set F_Γ of all $\alpha \in Q(\Gamma)_+$ such that $\langle \alpha, \varepsilon_i \rangle \leq 0$ for all i and the support of α is connected is called the *fundamental domain*. The elements of $\Delta_+^{im} = W(\Gamma)F_\Gamma \subseteq Q(\Gamma)_+$ are called *imaginary positive roots*. The positive root system $\Delta(\Gamma)_+$ is by definition the union of Δ_+^{re} and Δ_+^{im} .

Now we construct the root system of $\Gamma_{\pi,\sigma}$ in $Q(\Gamma)_{\pi,\sigma}$. For a vertex \bar{i} of $\Gamma_{\pi,\sigma}$, let $s_{\bar{i}} \in W(\Gamma)$ be $r_i r_{\pi(i)}$ if $i \notin \Gamma^\pi$ and r_i , otherwise. Note that $r_i r_{\pi(i)} = r_{\pi(i)} r_i$, because π is admissible. Let $W(\Gamma)_\pi$ be the subgroup in $W(\Gamma)$ generated by $s_{\bar{i}}$.

Proposition 6.4. *The lattice $Q(\Gamma)_{\pi,\sigma}$ is $W(\Gamma)_\pi$ -stable.*

Proof. It is sufficient to check $\gamma = s_{\bar{i}}(\alpha) \in Q(\Gamma)_{\pi,\sigma}$ for any $\alpha \in Q(\Gamma)_{\pi,\sigma}$. If $\pi(i) = i$, then $\gamma_j = \alpha_j, j \neq i$ and $\gamma_i = \alpha_i - 2\langle \alpha, \varepsilon_i \rangle$. Since i is adjacent to an even number of vertices, $\langle \alpha, \varepsilon_i \rangle$ is integer so γ_i is even whenever α_i is. If $\pi(i) \neq i$, then $s_{\bar{i}} = r_i r_{\pi(i)}$ and $\gamma_j = \alpha_j$ for any $j \neq i, j \neq \pi(i)$. Since α is symmetric, we have

$$(8) \quad \langle \alpha, \varepsilon_{\pi(i)} \rangle = \langle \alpha, \varepsilon_i \rangle.$$

Note also that $\langle \varepsilon_i, \varepsilon_{\pi(i)} \rangle = 0$, because π is admissible. Hence, $\langle r_{\pi(i)} \alpha, \varepsilon_i \rangle = \langle \alpha, \varepsilon_i \rangle$, so $\alpha_i - \gamma_i = \alpha_{\pi(i)} - \gamma_{\pi(i)} = 2\langle \alpha, \varepsilon_i \rangle$. Thus γ is symmetric whenever α is. \square

Starting from the graph $\Gamma_{\pi,\sigma}$ one can construct a symmetrizable GCM matrix $A(\Gamma_{\pi,\sigma})$ (see [Kac3, §4.7]). Namely, $A(\Gamma_{\pi,\sigma})_{ii} = 2$, $A(\Gamma_{\pi,\sigma})_{ij} = 0$ if i, j are not adjacent in $\Gamma_{\pi,\sigma}$, -1 for $i - j$ and for $i \Rightarrow j$, -2 for $i \Leftarrow j$. One can easily check:

Proposition 6.5. $A(\Gamma_{\pi,\sigma})_{ij} = \frac{2\langle \beta_i, \beta_j \rangle}{\langle \beta_i, \beta_i \rangle \langle \beta_j, \beta_j \rangle}$.

The root system $\Delta(\Gamma_{\pi,\sigma})$ is defined in terms of the matrix $A(\Gamma_{\pi,\sigma})$. Namely, we have a lattice $Q(\Gamma_{\pi,\sigma})$ generated by the elements of the simple root system $\Pi(\Gamma_{\pi,\sigma})$ and $A(\Gamma_{\pi,\sigma})$ gives rise to an integer linear form $\langle \beta', \alpha \rangle$ and a reflection $s'_{\beta'}(\alpha) = \alpha - \langle \beta', \alpha \rangle \beta'$ on it for each $\beta' \in \Pi(\Gamma_{\pi,\sigma})$. The Weyl group $W(\Gamma_{\pi,\sigma})$ is generated by these reflections. Comparing the above results, we get:

Corollary 6.6. *The natural bijection $\beta_i \rightarrow \beta'_i$ of $\Pi_{\pi,\sigma}$ onto $\Pi(\Gamma_{\pi,\sigma})$ gives rise to an isomorphism of groups $W(\Gamma)_\pi \cong W(\Gamma_{\pi,\sigma})$ and an equivariant bijection of $(Q(\Gamma)_{\pi,\sigma}, W(\Gamma)_\pi)$ onto $(Q(\Gamma_{\pi,\sigma}), W(\Gamma_{\pi,\sigma}))$.*

Observe that by formula (8) the condition $\langle \alpha, \varepsilon_i \rangle \leq 0$ for all i is equivalent for $\alpha \in Q(\Gamma)_{\pi, \sigma}$ to another system of inequalities: $\langle \alpha, \beta_j \rangle \leq 0$ for all j . Therefore the set $F_{\Gamma, \pi, \sigma}$ of all $\alpha \in Q(\Gamma)_{\pi, \sigma, +}$ having a connected support and meeting the above inequalities is equal to $F_{\Gamma} \cap \Gamma_{\pi, \sigma}$. Thus we have:

Corollary 6.7. *The positive root system $\Delta(\Gamma_{\pi, \sigma})_+$ is $\Delta(\Gamma_{\pi, \sigma})_+^{re} \cup \Delta(\Gamma_{\pi, \sigma})_+^{im}$, where*

$$(9) \quad \Delta(\Gamma_{\pi, \sigma})_+^{re} = W(\Gamma)_{\pi} \Pi_{\pi, \sigma} \cap Q(\Gamma)_+, \quad \Delta(\Gamma_{\pi, \sigma})_+^{im} = W(\Gamma)_{\pi} F_{\Gamma, \pi, \sigma}.$$

We now describe the root system $\Delta(\Gamma_{\pi, \sigma})_+$ in terms of $\Delta(\Gamma)$. For any $\alpha \in Q(\Gamma)$ set $\bar{\alpha} = \alpha$, if $\alpha \in Q(\Gamma)_{\pi, \sigma}$, and $\bar{\alpha} = \alpha + \pi(\alpha)$, otherwise. The subsequent Lemma and Proposition in the case when $\sigma(i) = 1$ for all $i \in \Gamma^{\pi}$ follow from [Hu, Proposition 4].

Lemma 6.8. $\Delta(\Gamma_{\pi, \sigma})_+ = \overline{\Delta(\Gamma)_+} = \{\bar{\alpha} | \alpha \in \Delta(\Gamma)_+\}.$

Proof. We know that $\Delta(\Gamma_{\pi, \sigma})_+ = \Delta(\Gamma_{\pi, \sigma}) \cap Q(\Gamma)_+$ and analogously $\overline{\Delta(\Gamma)_+} = \overline{\Delta(\Gamma) \cap Q(\Gamma)_+}$. So it is sufficient to show $\Delta(\Gamma_{\pi, \sigma}) = \overline{\Delta(\Gamma)}$. By Proposition 6.4 and since the action of $W(\Gamma)_{\pi}$ commutes with π , both sets are $W(\Gamma)_{\pi}$ -stable. On the other hand, $\Pi_{\pi, \sigma}$ and $F_{\Gamma, \pi, \sigma}$ are contained in $\overline{\Delta(\Gamma)_+}$. Hence, Corollary 6.7 yields the inclusion \subseteq in the Lemma.

Conversely, assume $\gamma \in \overline{\Delta(\Gamma)_+}$. Apply induction on $\text{ht} \gamma = \gamma_1 + \dots + \gamma_n$ to prove that γ belongs to $\Delta(\Gamma_{\pi, \sigma})_+$. We have three possible cases:

1. For some i , $s_i(\gamma) \in Q(\Gamma)_+$, $\langle \beta_i, \gamma \rangle > 0$. In this case $\text{ht}_{s_i}(\gamma) < \text{ht} \gamma$ and $s_i(\gamma) \in \overline{\Delta(\Gamma)_+}$. Hence, we are done by induction.
2. $\gamma \in F_{\Gamma, \pi, \sigma}$. Clearly, in this case we are done.
3. For some i , $s_i(\gamma) \notin Q(\Gamma)_+$. It is well-known that for a root α of Γ and a simple reflection $r_i \in W(\Gamma)$, $r_i \alpha \notin Q(\Gamma)_+$ is equivalent to $\alpha = \varepsilon_i$. Applying this observation, we get $\gamma = \bar{\varepsilon}_i \in \Delta(\Gamma_{\pi, \sigma})_+$. This completes the proof. \square

Proposition 6.9. *If $\bar{\alpha} \in \Delta(\Gamma_{\pi, \sigma})_+^{re}$, then $\alpha \in \Delta(\Gamma)^{re}$ and $\bar{\alpha'} \neq \bar{\alpha}$ for $\alpha' \neq \alpha, \pi(\alpha)$.*

Proof. First note that $\Delta(\Gamma_{\pi, \sigma})_+^{re}$ is divided into 3 disjoint set (possibly empty):

$$(10) \quad \Delta(\Gamma_{\pi, \sigma})_+^{re} = \bigcup_{\sigma(i)=1} W(\Gamma)_{\pi} \varepsilon_i \sqcup \bigcup_{j < \pi(j)} W(\Gamma)_{\pi} (\varepsilon_j + \varepsilon_{\pi(j)}) \sqcup \bigcup_{\sigma(k)=-1} W(\Gamma)_{\pi} 2\varepsilon_k.$$

Since the action of $W(\Gamma)$ preserves the scalar product we have $\langle \gamma, \gamma \rangle$ is 1, 2, 4 for γ from the first, second, and third sets, respectively, because $\langle \varepsilon_j, \varepsilon_{\pi(j)} \rangle = 0$.

If α is an imaginary root, then the elements of the smallest height in the $W(\Gamma)_{\pi}$ -orbit of $\bar{\alpha}$ belong to $F_{\Gamma, \pi, \sigma}$ so $\bar{\alpha}$ cannot be real.

Assume that $\gamma = \bar{\alpha}_1 = \bar{\alpha}_2$ for two real roots α_1 and α_2 . If γ belongs to the first subset in formula (10), then $\gamma = \alpha_1 = \alpha_2$. Analogously, assuming γ to be in the third subset, we get $\gamma = 2\alpha_1 = 2\alpha_2$.

Now assume $\gamma = w\varepsilon_j + w\varepsilon_{\pi(j)} = \alpha + \pi(\alpha)$. Note that for any s_i , and any $\delta \in \Delta(\Gamma)_+$, $s_i(\delta + \pi(\delta)) \in Q(\Gamma)_+$ implies $s_i \delta \in Q(\Gamma)_+$. Then $\beta = w^{-1}\alpha \in Q(\Gamma)_+$ and $\varepsilon_j + \varepsilon_{\pi(j)} = \beta + \pi(\beta)$. Thus β is either ε_j or $\varepsilon_{\pi(j)}$. This completes the proof. \square

Assume that a signed quiver Q^{σ} fulfills the condition of 6.2; denote by Γ the underlying graph of \tilde{Q} endowed with an admissible involution $\pi = *$.

Corollary 6.10. *Suppose that for any $\alpha \in \Delta(\Gamma)_+ \cap Q(\Gamma)_{\pi, \sigma}$ either an indecomposable representation of dimension α is symmetric or there is an indecomposable*

representation of dimension $\alpha/2$, which is not isomorphic to any symmetric one. Then:

1. The set of dimensions of indecomposable symmetric representations of \tilde{Q} is equal to $\Delta(\Gamma_{\pi,\sigma})_+$.

2. If $\gamma \in \Delta(\Gamma_{\pi,\sigma})_+^{re}$, then there exists a unique indecomposable symmetric representation of dimension γ modulo isomorphism.

Proof. 1. By Proposition 6.8 we may and will replace $\Delta(\Gamma_{\pi,\sigma})_+$ by $\overline{\Delta(\Gamma)}_+$. If X is an indecomposable symmetric representation, then $\dim X$ is sign-matched, i.e., belongs to $Q(\Gamma)_{\pi,\sigma}$. By Lemma 4.5, either X is indecomposable as a representation of \tilde{Q} and in this case $\dim X = \dim \bar{X} \in \overline{\Delta(\Gamma)}_+$; or $X = V + V^*$ and $\dim X = \alpha + \pi(\alpha)$, $\alpha = \dim V$. In the latter case, if α is not sign-matched, then $\dim X = \bar{\alpha} \in \overline{\Delta(\Gamma)}_+$. Assume that α is sign-matched; if α is real, then $\alpha/2 \notin \Delta(\Gamma)$ and by the hypothesis there exists an indecomposable symmetric representation V' of dimension α . However, V' must be isomorphic to V , since α is a real root. We therefore get a contradiction with Proposition 4.6. Thus α is sign-matched and imaginary implying that $\dim X = 2\alpha$ is also an imaginary root, hence, $\dim X = \dim \bar{X} \in \overline{\Delta(\Gamma)}_+$.

Conversely, if $\gamma \in \overline{\Delta(\Gamma)}_+$, then either $\gamma = \alpha + \pi(\alpha)$ with α being a non-sign-matched root or γ itself is a sign-matched root. In the former case for any indecomposable representation V of dimension α , V is not symmetric, hence, $V + V^*$ is indecomposable symmetric of dimension γ . In the latter case the hypothesis yields γ is the dimension of an indecomposable symmetric representation.

2. By Proposition 6.9, $\gamma = \bar{\alpha}$ for a unique (modulo π) root α , and besides α is a real root. The fact that all indecomposable representations of dimension α are conjugate to each other and Corollary 3.18 complete the proof. \square

7. SIGNED QUIVERS OF FINITE TYPE.

Definition 7.1. We call a signed quiver Q^σ *finite* if there are finitely many isomorphism classes of indecomposable symmetric representations of \tilde{Q} .

Note that any usual quiver S gives rise to a signed quiver S^σ with $\sigma = 0$ for all vertices and arrows and no vertices-twins. Clearly a signed quiver is finite if and only if each connected component is finite.

Theorem 7.2. For a connected signed quiver Q^σ 3 conditions are equivalent:

- (i) Q^σ is finite
- (ii) \tilde{Q} is finite
- (iii) Q^σ is either a finite quiver or is one of the following signed quivers:

$$B_n : \circ - \dots - \circ - \overset{+}{\circ} \quad B_{n,+} : \circ - \dots - \circ - \overset{+}{\circ} \circ$$

$$C_n : \circ - \dots - \circ - \bar{\circ} \quad C_{n,-} : \circ - \dots - \circ - \bar{\circ} \circ$$

where the orientation of edges is arbitrary, n is the number of vertices.

Remark 7.1. An analogous result for generalized quivers is proved in [DW, 3.1].

Proof. The implication (iii) \Rightarrow (ii) is obvious. To prove (ii) \Rightarrow (iii) first observe that if Q^σ is not a usual quiver, then \tilde{Q} is connected and admits a nontrivial

involutive automorphism. Since \tilde{Q} is finite, it is of type A_m, D_m, E_6 . The involutive automorphisms of D_m and E_6 have a pair of fixed vertices connected by a unique arrow; this is impossible for \tilde{Q} , because by Definition 3.1 we have for this arrow φ : $t(\varphi^*) = h\varphi, h(\varphi^*) = t\varphi$. So \tilde{Q} is A_m with the central symmetry as $*$ and Q is $B_{(m+1)/2}, C_{(m+1)/2}$ if m is odd and $B_{m/2+1,+}, C_{m/2+1,-}$ otherwise.

The implication (ii) \Rightarrow (i) follows directly from Lemma 4.5. Conversely, assume \tilde{Q} is not finite. Then for an imaginary root α there exists an infinite family $V(\lambda)$ of pairwise non-isomorphic indecomposable representations of dimension α . Then either of two families, $V(\lambda)$ and $V(\lambda) + V(\lambda)^*$ contains infinitely many non-isomorphic indecomposable symmetric representations. \square

We want to describe explicitly the indecomposable symmetric representations of the finite signed quivers. Note that Proposition 5.9 reduces this question for $B_{n,+}$ to that for B_n . Indeed, we have $B_n = \text{Ins}_\varphi(B_{n,+})$, where φ denote the signed arrow of $B_{n,+}$. Analogously, $C_n = \text{Ins}_\varphi(C_{n,-})$ and it is sufficient to consider B_n and C_n .

By Lemma 4.5, to describe the indecomposable symmetric representations of Q^σ we only need to answer a question as follows: which indecomposable representations V of \tilde{Q} admit a signed form making V symmetric? Clearly, a necessary condition is that $\dim V$ is sign-matched. In both cases $\tilde{Q} = A_{2n+1}$ and the indecomposable representations have dimensions of form $(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)$. So for C_n none root is sign-matched, because the middle dimension is odd. For B_n the symmetric dimensions are sign-matched and clearly, the corresponding indecomposable representations are symmetric with respect to an appropriate signed form. Note that for both B_n and C_n the involution $*$ is admissible and the above description shows that the hypothesis of Corollary 6.10 holds. Note also that the graph $\Gamma_{\pi,\sigma}$ is the Dynkin diagram B_n and C_n for the quiver B_n and C_n , respectively. Hence, we obtain:

Theorem 7.3. *The set of dimensions of indecomposable symmetric representations of B_n and C_n constitute a root system of type B_n and C_n , respectively.*

Remark 7.2. An analogous result for generalized quivers is proved in [DW, 3.6].

8. SIGNED QUIVERS OF TAME TYPE.

Definition 8.1. We call a signed quiver Q^σ *tame* if infinite families of classes of indecomposable symmetric representations of \tilde{Q} exist and depend on 1 parameter.

Theorem 8.2. *For a connected signed quiver Q^σ 3 conditions are equivalent:*

- (i) Q^σ is tame
- (ii) \tilde{Q} is tame
- (iii) Q^σ is either a tame quiver or is one of the following signed quivers:

$$\begin{aligned}
 (11) \quad & O_+ : \overset{\circ}{\circ}^+ + \quad O_- : \overset{\circ}{\circ}^+ - \quad Sp_- : \overset{\circ}{\circ}^- - \quad Sp_+ : \overset{\circ}{\circ}^- + \\
 & D_n^{(2)} : \overset{+}{\circ} - \circ - \dots - \circ - \overset{+}{\circ} \quad D_{n,+}^{(2)} : \overset{+}{\circ} - \circ - \dots - \circ - \overset{+}{\circ} \quad D_{n,++}^{(2)} : \overset{+}{\circ} - \circ - \dots - \circ - \overset{+}{\circ} \\
 & C_n^{(1)} : \overset{-}{\circ} - \circ - \dots - \circ - \overset{-}{\circ} \quad C_{n,-}^{(1)} : \overset{-}{\circ} - \circ - \dots - \circ - \overset{-}{\circ} \quad C_{n,--}^{(1)} : \overset{-}{\circ} - \circ - \dots - \circ - \overset{-}{\circ}
 \end{aligned}$$

$$\begin{array}{c}
A_{2n}^{(2)} : \bar{\circ} - \circ - \dots - \circ - \overset{+}{\circ} \\
\\
A_{2n,+}^{(2)} : \bar{\circ} - \circ - \dots - \circ - \overset{+}{\circ} \quad A_{2n,-}^{(2)} : \circ - \bar{\circ} - \dots - \circ - \overset{+}{\circ} \quad A_{2n,+-}^{(2)} : \circ - \bar{\circ} - \dots - \circ - \overset{+}{\circ} \\
\\
Z_n : \circ_1 - \circ - \dots - \circ - \circ_{1*} \\
\\
\begin{array}{cc}
B_n^{(1)} : \overset{+}{\circ} - \circ - \dots - \circ - \circ & B_{n,+}^{(1)} : \circ - \overset{+}{\circ} - \dots - \circ - \circ \\
\downarrow & \downarrow \\
\circ & \circ
\end{array} \\
\\
\begin{array}{cc}
A_{2n-1}^{(2)} : \bar{\circ} - \circ - \dots - \circ - \circ & A_{2n-1,-}^{(2)} : \circ - \bar{\circ} - \dots - \circ - \circ \\
\downarrow & \downarrow \\
\circ & \circ
\end{array}
\end{array}$$

where the orientation of edges is arbitrary, the number of vertices is $n + 1$.

Remark 8.1. A similar result for generalized quivers is proved in [DW, 4.2]. However, the signed quivers $O_+, Sp_-, D_{n,+}^{(2)}, C_{n,-}^{(1)}, A_{2n}^{(2)}, A_{2n,+-}^{(2)}$ have no counterpart in that theory (see Remark 2.1).

Proof. We argue exactly as in the proof of 7.2. In the proof of implication (ii) \Rightarrow (iii) we have 4 possibilities for the underlying graph Γ of \tilde{Q} and the involutive automorphism π of Γ :

- (a) : Γ is a graph with 1 vertex and one loop, π is trivial;
- (b) : Γ is a circular graph $A_m^{(1)}$ ($m + 1$ vertices), π is a mirror symmetry;
- (c) : Γ is a circular graph $A_m^{(1)}$, m is odd, π is the central symmetry;
- (d) : Γ is $D_n^{(1)}$, π is the central symmetry.

The case (a) yields the signed quivers from (11). In the case (b) there are several possibilities for the mirror: if the mirror meets two vertices, we get $D_n^{(2)}, C_n^{(1)}, A_{2n}^{(2)}$; if the mirror meets a vertex and the opposite edge, we get $D_{n,+}^{(2)}, C_{n,-}^{(1)}, A_{2n,+}^{(2)}, A_{2n,-}^{(2)}$; finally, when the mirror meets two edges, we get $D_{n,++}^{(2)}, C_{n,--}^{(1)}, A_{2n,+-}^{(2)}$. The case (c) yields $Z_{(m+1)/2}$. The case (d) with center being a vertex yields $B_n^{(1)}, A_{2n-1}^{(2)}$ and with center being on an edge yields $B_{n,+}^{(1)}, A_{2n-1,-}^{(2)}$. \square

We are now going to describe the indecomposable symmetric representations for the tame signed quivers. Note that the notation introduced in 8.2 divide them into families with a base term X and additional terms X_+, X_- etc. For example, $A_{2n}^{(2)}$ has 3 relatives: $A_{2n,+}^{(2)}, A_{2n,-}^{(2)}, A_{2n,+-}^{(2)}$. These quivers can be obtained from each other applying repeatedly the functors Del and Ins: $\text{Ins}_\varphi(A_{2n,+-}^{(2)}) = A_{2n,-}^{(2)}$, $\text{Ins}_\psi(A_{2n,+-}^{(2)}) = A_{2n,+}^{(2)}$, $\text{Ins}_\varphi(A_{2n,+}^{(2)}) = A_{2n}^{(2)} = \text{Ins}_\psi(A_{2n,-}^{(2)})$, where φ and ψ are the $+$ and the $-$ arrow of $A_{2n,+-}^{(2)}$, respectively. So if we know the indecomposable symmetric representations for $A_{2n}^{(2)}$, Proposition 5.9 yields the same

for other 3 quivers. We therefore need to consider the quivers from (11) and $D_n^{(2)}, C_n^{(1)}, A_{2n}^{(2)}, Z_n, B_n^{(1)}, A_{2n-1}^{(2)}$.

By Lemma 4.5, to describe the indecomposable symmetric representations of Q^σ is equivalent to find the indecomposable representations V of \tilde{Q} such that $\dim V$ is sign-matched and V is symmetric with respect to an appropriate signed form. First we consider the quivers from (11). Here \tilde{Q} is the quiver with one arrow-loop, an indecomposable representation A is just a Jordan matrix $J_n(\lambda)$.

Example 8.3. $Q^\sigma = O_+$. A signed form for A is a symmetric matrix J such that JA is symmetric (cf. 3.9).

Proposition 8.4. *Any matrix A has a presentation $JA = B$ with $J^\top = J$, $\det J \neq 0$, $B^\top = B$.*

Proof. We can rewrite $A = CB$ with $C = J^{-1}, C^\top = C$. Furthermore, a standard fact from Linear Algebra is that C can be presented as $C = DD^\top$, hence, $A = DD^\top BDD^{-1}$. Our presentation is therefore equivalent to the classical fact that A is conjugate to a symmetric matrix $D^\top BD$ (see. e.g. [Ga]). \square

Example 8.5. $Q^\sigma = O_-$. A signed form for A is a symmetric matrix J such that JA is anti-symmetric (cf. 3.8). Arguing as in the proof of 8.4, we get:

Proposition 8.6. *Jordan matrix $A = J_n(\lambda)$ has a presentation $JA = B$ with $J^\top = J$, $\det J \neq 0$, $B^\top = -B$ if and only if $\lambda = 0$ and n is odd.*

Example 8.7. $Q^\sigma = Sp_+$. We consider only representations of sign-matched dimensions, so n is even. A signed form for A is an anti-symmetric matrix J such that JA is symmetric.

Proposition 8.8. *Jordan matrix $A = J_n(\lambda)$ has a presentation $JA = B$ with $J^\top = -J$, $\det J \neq 0$, $B^\top = B$ if and only if $\lambda = 0$ and n is even.*

Proof. The property is equivalent to $A = CB$ with $C^\top = -C$, $\det C \neq 0$. Using the properties of trace, we get: $\text{tr}(A) = \text{tr}(CB)^\top = -\text{tr}(BC) = -\text{tr}(A)$. Hence, $\text{tr}(A) = n\lambda = 0$. In addition, $\det C \neq 0$ implies n is even. Conversely, given an even n we take C' and B' such that $C'_{ij} = 0$ for $i + j \neq n + 1$ and $B'_{ij} = 0$ for $i + j \neq n + 2$. Then $C'B'$ is conjugate to $J_n(0)$, hence, $J_n(0)$ can also be presented in this form. \square

Example 8.9. $Q^\sigma = Sp_-$. Again n is even. A signed form for A is an anti-symmetric matrix J such that JA is anti-symmetric.

Proposition 8.10. *None Jordan matrix $A = J_n(\lambda)$ with even n has a presentation $JA = B$ with $J^\top = -J$, $\det J \neq 0$, $B^\top = -B$.*

Proof. We have: $B = J(\lambda \text{Id} + J_n(0))$ and $B - \lambda J$ is anti-symmetric, so we may assume $\lambda = 0, B = JJ_n(0)$. However, i -th column of $J_n(0)$ is $i - 1$ -th column of J for $i > 1$ and the first one is zero. Using the condition that both J and B are anti-symmetric, we get $B = J = 0$. \square

Now we consider $D_n^{(2)}, C_n^{(1)}, A_{2n}^{(2)}, B_n^{(1)}, A_{2n-1}^{(2)}$. Observe that all these quivers with $n \geq 2$ fulfill the condition from 6.2 and the involution π of the underlying graph Γ of \tilde{Q} is admissible. As diagram $\Gamma_{\pi, \sigma}$ we obtain the affine graph denoted by exactly the same term as the quiver: $D_n^{(2)}$ for $D_n^{(2)}$ etc. (it is of course more

correct to say that we denoted our signed quivers following the notation of Kac, see [Kac2],[Kac3]). For the signed quiver $Z_n, n \geq 2$ the involution is also admissible and we have $\Gamma_{\pi,\sigma} = A_{n-1}^{(1)}$. Below we consider these quivers case by case and prove

Theorem 8.11. *Let Q^σ be one of the quivers $D_n^{(2)}, C_n^{(1)}, A_{2n}^{(2)}, Z_n, B_n^{(1)}, A_{2n-1}^{(2)}, n \geq 2$. Then the set of dimensions for the indecomposable symmetric representations of \tilde{Q} is equal to the root system $\Delta(\Gamma_{\pi,\sigma})_+$ of the corresponding affine graph $\Gamma_{\pi,\sigma}$. Furthermore, all indecomposable symmetric representations with dimension from $\Delta(\Gamma_{\pi,\sigma})_+^{re}$ are isomorphic to each other.*

Proof. For almost all cases the Theorem follows from Corollary 6.10 and

Proposition 8.12. *The hypothesis of 6.10 holds for the quivers $D_n^{(2)}, C_n^{(1)}, A_{2n}^{(2)}, B_n^{(1)}, A_{2n-1}^{(2)}, n \geq 2$, i.e., for any sign-matched root α either an indecomposable representation of dimension α is symmetric or there is an indecomposable representation of dimension $\alpha/2$, which is not isomorphic to any symmetric one.*

Proof. (of Proposition) For $Q^\sigma = D_n^{(2)}, C_n^{(1)}, A_{2n}^{(2)}$ the underlying graph of \tilde{Q} is $A_{2n-1}^{(1)}$ such that the vertices in the clockwise order are $1, 2, \dots, n+1, n^*, \dots, 2^*$. For any $i \in \tilde{Q}_0$, write $i+1$ and $i-1$ for the next and the previous vertex in the clockwise order, respectively; denote by $\varphi_i \in \tilde{Q}_1$ the unique arrow such that $\{t\varphi_i, h\varphi_i\} = \{i, i+1\}$. We keep notation $J_d(\lambda)$ for the corresponding Jordan matrix; by $I_{r,s}$ we denote the $s \times r$ matrix having the first $p = \min(r, s)$ standard basis vectors of \mathbf{k}^s as the first p columns, the last $r-p$ columns being zero; by $I'_{r,s}$ we denote the $s \times r$ matrix having the last p standard basis vectors of \mathbf{k}^s as the last p columns, the first $r-p$ columns being zero; for the identity matrix $I_{d,d}$ we just write I_d .

The indecomposable representations of \tilde{Q} are as follows. The root $\delta = (1, \dots, 1)$ generates the fundamental domain $F_{\tilde{Q}}$. For any natural d we have a 1-dimensional family $V_\lambda^d, \lambda \in \mathbf{k}$ of indecomposable representations of dimension $d\delta$:

$$(12) \quad V_\lambda^d(\varphi_1) = J_d(\lambda), V_\lambda^d(\varphi_i) = I_d, i \neq 1.$$

The real roots are the vectors $\alpha^{k,l,d}$ such that: $(\alpha^{k,l,d} - d\delta)_i = 1$, for $i = k, k+1, \dots, l$ and 0 for other vertices, $k, l \in \tilde{Q}_0, d \in \mathbf{Z}_+$. For each $\alpha^{k,l,d}$ we have an indecomposable representation $V^{k,l,d}$ as follows:

$$(13) \quad V^{k,l,d}(\varphi) = I_{r,s}, \varphi \in \tilde{Q}_1 \setminus \{\varphi_l\}, V^{k,l,d}(\varphi_l) = I'_{r,s}, r = \alpha_{t\varphi}^{k,l,d}, s = \alpha_{h\varphi}^{k,l,d}.$$

Note that $V^{k,l,d}(\varphi_i)$ is the identity matrix for all i but l and $k-1$.

1. $Q^\sigma = D_n^{(2)}$. The sign-matched dimension vectors are just symmetric ones; so the sign matched roots are $d\delta$ and $\alpha^{i,i^*,d}$ for arbitrary i and d . We claim that the corresponding representations V_λ^d and $V^{i,i^*,d}$ are symmetric with respect to an appropriate signed form.

Let $V = V_\lambda^d$. Then in the dual bases of $V(i)^*$ the representation V^* is given by the matrices as follows: $V^*(\varphi) = I_d$ for $\varphi \neq \varphi_{2^*}$, and $V^*(\varphi_{2^*}) = J_d^\top(\lambda)$. We are looking for a signed form $J_k : V(k) \rightarrow V(k^*)^*$ such that $J_1^* = J_1, J_{n+1}^* = J_{n+1}, J_i^* = J_{i^*}$ for $i = 2, \dots, n$, and J is an isomorphism of V onto V^* . Taking into account the explicit form of V and V^* we rewrite the condition: $J_2 = J_3 = \dots =$

$J_{n+1} = J_n^* = \cdots = J_2^*$, $J_1^\top = J_1$, $J_2^\top = J_2$, and the diagram commutes:

$$\begin{array}{ccccc} V(2^*) & \xrightarrow{I_d} & V(1) & \xrightarrow{J_d(\lambda)} & V(2) \\ J_2^\top \downarrow & & \downarrow J_1 & & \downarrow J_2 \\ V(2)^* & \xrightarrow{J_d^\top(\lambda)} & V(1)^* & \xrightarrow{I_d} & V(2^*)^* \end{array}$$

The left square condition is equivalent to the right one: $J_2 J_d(\lambda) = J_1$. By Proposition 8.4, such J_1 and J_2 exist.

Let $V = V^{i,i^*,d}$, $\alpha = \alpha^{i,i^*,d}$. Note that $(i-1)^* = i^* + 1$ and $\varphi_{i-1}^* = \varphi_{i^*}$. So $V(\varphi)$ is the identity matrix for all arrows but $\psi = \varphi_{i-1}$ and ψ^* , $V(\psi) = I_{r,s}$, $V(\psi^*) = I'_{s,r}$, where $r = \alpha_{t\psi} = \alpha_{h\psi^*}$, $s = \alpha_{h\psi} = \alpha_{t\psi^*}$. Hence, $V^*(\varphi) = I_d$ or I_{d+1} for $\varphi \neq \psi, \psi^*$, $V^*(\psi) = I'_{r,s}$, $V^*(\psi^*) = I_{s,r}$. So the condition on the signed form J_k is equivalent to finding a symmetric $d \times d$ matrix J_1 and symmetric $d+1 \times d+1$ matrix J_2 such that $I'_{d,d+1} J_1 = J_2 I_{d,d+1}$. We may take both J_1 and J_2 having units on the secondary diagonal and 0 outside it. Thus the Proposition is proved for this case.

2. $Q^\sigma = C_n^{(1)}$. A sign-matched dimension vector α is a symmetric one such that α_1 and α_{n+1} are both even. So the only sign-matched roots are $2m\delta$ for arbitrary m . We claim that none representation V_λ^d is symmetric; clearly, this implies what we need. Assume that d is even, otherwise the claim is obvious. For $V = V_\lambda^d$, V^* is as in case **1** but for two arrows with heads $1, n+1$ the maps are multiplied by -1 (see Definition 3.3). Similarly to case **1**, a signed form can be reduced to anti-symmetric non-degenerate $d \times d$ matrices J_1, J_2 such that $J_2 J_d(\lambda) = J_1$. By Proposition 8.10, this is impossible.

3. $Q^\sigma = A_{2n}^{(2)}$. A sign-matched dimension vector α is a symmetric one such that α_1 is even. So we have more possibilities for α : $2m\delta$ and $\alpha^{i,i^*,d}$, where either d is even and $i = 2, \dots, n+1$, or d is odd and $i = n^*, \dots, 2^*, 1$. We claim that there are no symmetric representations isomorphic to V_λ^{2m} , whereas for the real sign-matched roots a symmetric representation exists.

Let $V = V_\lambda^{2m}$. Then, as in cases **1,2** a signed form can be reduced to non-degenerate $d \times d$ matrices J_1, J_2 , such that $J_1^\top = -J_1$, $J_2^\top = J_2$, and $J_2 J_{2m}(\lambda) = J_1$. By Proposition 8.6, this is impossible.

Let $V = V^{i,i^*,d}$, $\alpha = \alpha^{i,i^*,d}$, where d is even and $i = 2, \dots, n+1$. As in case **1** a signed form can be reduced to an anti-symmetric $d \times d$ matrix J_1 and a symmetric $d+1 \times d+1$ matrix J_2 such that $I'_{d,d+1} J_1 = J_2 I_{d,d+1}$. Take J_1 having a sequence $1, -1, 1, \dots, -1$ on the secondary diagonal and 0 outside it; as J_2 take the submatrix of J_1 obtained removing first row and last column. The case when d is odd and $i = n^*, \dots, 1^*, 1$ is similar.

For the quivers $Q^\sigma = B_n^{(1)}, A_{2n-1}^{(2)}$ the underlying graph of \tilde{Q} is $D_{2n}^{(1)}$ with vertices $1, 2, 3, \dots, n, n+1, n^*, \dots, 3^*, 2^*, 1^*$. Assume that $1, 2$ are tails of the arrows $\chi, \psi \in \tilde{Q}_1$ respectively such that $h\chi = h\psi = 3$ (this is possible not for each orientation but the consideration of other cases is similar).

The symmetric roots are as follows. The root $\delta = (1, 1, 2, \dots, 2, 1, 1)$ generates the fundamental domain $F_{\tilde{Q}}$. The real symmetric roots are of form $d\delta + \alpha$, where either $d \geq 0$, each α_i is 0 or 1 and the support of α is a connected symmetric subgraph in $D_{2n}^{(1)}$, or $d > 0$ and $\alpha = -\varepsilon_i - \dots - \varepsilon_{i^*}$ with $i = 4, \dots, n+1$. In dimension

$d\delta$ we construct a 1-parameter family V_λ^d of indecomposable representations:

$$(14) \quad V_\lambda^d(\chi) = (I_d|0)^\top, V_\lambda^d(\psi) = (0|I_d)^\top, V_\lambda^d(\chi^*) = (I_d|I_d), V_\lambda^d(\psi^*) = (I_d|J_d(\lambda)),$$

$V_\lambda^d(\varphi) = I_{2d}$ for all other arrows φ .

4. Let $Q^\sigma = B_n^{(1)}$. Here all the above symmetric roots are sign-matched. We claim that all the corresponding indecomposable representations are symmetric with respect to an appropriate signed form.

Let $V = V_\lambda^d$. Then in the dual bases of $V(i)^*$ the representation V^* is:

$$(15) \quad V^*(\chi) = (I_d|I_d)^\top, V^*(\psi) = (I_d|J_d(\lambda))^\top, V^*(\chi^*) = (I_d|0), V^*(\psi^*) = (0|I_d),$$

$V_\lambda^d(\varphi) = I_{2d}$ for all other arrows φ . For a signed form $J \in \text{Hom}(V, V^*)$ we therefore have: $J_3 = J_4 = \dots = J_{3^*} = A$, where A is a symmetric $2d \times 2d$ matrix. Denote by $A_{11} = A_{11}^\top, A_{12}, A_{21} = A_{12}^\top, A_{22} = A_{22}^\top$ the $d \times d$ blocks of A . Then a direct calculation shows that J is a signed form for V if and only if: $J_1 = J_2 = A_{11} = A_{12} = A_{21}$ and $A_{22} = J_d(\lambda)A_{11}$. By Proposition 8.4, we may find such symmetric matrices A_{11} and A_{22} .

As for the real roots, we will not present calculations for all of them, because the notation needs much place. We just consider the case $n = 2, \alpha = (0, 0, 1, 0, 0)$ and we assure a reader that the other cases are not too different. As an indecomposable representation of dimension $d, d, 2d+1, d, d$ we can take V such that:

$$(16) \quad V(\chi) = (I_d|0)^\top, V(\psi) = (I_d|I_{d+1,d})^\top, V(\chi^*) = (I_d|0), V(\psi^*) = (I_d|I'_{d+1,d}).$$

Let a collection $(J_1, J_2, A = J_3, J_1^\top, J_2^\top)$ be a candidate for a signed form. Decompose A into blocks with respect to the decomposition of $\mathbf{k}^{2d+1} = \mathbf{k}^d \oplus \mathbf{k}^{d+1}$ used in (16): $A_{11} = A_{11}^\top, A_{12}, A_{21} = A_{12}^\top, A_{22} = A_{22}^\top$. Then a direct calculation shows that J is a signed form for V if and only if: $J_1 = J_2 = A_{11}, A_{12} = A_{21} = 0$, and $I'_{d,d+1}A_{11} = A_{22}I_{d,d+1}$. We indicated the appropriate A_{11} and A_{22} in case 1.

5. Let $Q^\sigma = A_{2n-1}^{(2)}$. Here a symmetric vector α is sign-matched if and only if α_{n+1} is even. Hence, the only sign-matched roots are the multiples of δ . We claim that none representation $V = V_\lambda^d$ is symmetric with respect to a signed form. Formula (15) holds for V^* and for all other arrows $V^*(\varphi) = I_{2d}$ except for the arrow ρ having $n+1$ as the head, where $V^*(\rho) = -I_{2d}$. Arguing as in 4 we reduce a signed form to a presentation $A_{22} = J_d(\lambda)A_{11}$, where A_{11} and A_{22} are anti-symmetric non-degenerate matrices. By Proposition 8.10, such a presentation does not exist.

Thus we considered all 5 signed quivers and the Proposition is proved. \square

To finish the proof of the Theorem we need to consider the quiver $Q^\sigma = Z_n$. In this case, $\tilde{Q} = A_{2n-1}^{(1)}$ such that the vertices in the clockwise order are $1, 2, \dots, n, 1^*, 2^*, \dots, n^*$. The involution π takes each vertex to its opposite and each symmetric dimension vector is sign-matched. The roots of type $\alpha^{k,l,d}$ are therefore not symmetric, so the only sign-matched roots are the multiples of δ .

Let $V = V_\lambda^d$. We claim: V is symmetric with respect to a signed form if and only if either $\lambda = 1$ and d is odd or $\lambda = -1$ and d is even. Actually, arguing as in the proof of 8.12, we see that a signed form can be reduced to a non-degenerate matrix J such that $JJ_d(\lambda) = J^\top$. Set $A = J_d(\lambda)$ and rewrite the condition as $A = J^{-1}J^\top$; then we get $A^{-\top} = J^\top J^{-1}$, hence, $\text{tr}(A) = \text{tr}(A^{-\top})$ and $\lambda = \lambda^{-1}$. So we have:

$$(17) \quad \lambda J + JJ_d(0) = J^\top, \lambda = \pm 1, \det J \neq 0.$$

If $\lambda = 1$, this implies $d - 1 = \text{rank} J J_d(0)$ is even, since $J^\top - J$ is anti-symmetric. If $\lambda = -1$, one can show that any solution of (17) has the properties: $J_{ij} = 0$, if $i + j \leq d$, $J_{ij} = -J_{ji}$, if $i + j = d + 1$; so $\det J \neq 0$ implies d is even. A reader can check that under the above restrictions (17) has a solution.

Now we see that the assertion of 6.10 is not true only for the imaginary roots $\gamma = (2m + 1)\delta$. For this case we can however present γ as the symmetrization $\alpha + \bar{\alpha}$ of a real root α such that $\alpha_i = m + 1$, $\alpha_{j^*} = m$. Thus the proof of 6.10 yields the assertion of the Theorem for Z_n . \square

REFERENCES

- [DW] Derksen, H., and Weyman, J.: Generalized quivers associated to reductive groups, *Colloq. Math.* **94**, 2 (2002), 151-173.
 - [Ga] Gantmacher, F.R.: *Matrix theory*. Springer, Berlin, 2nd edition, 1986.
 - [Hu] Hubery, A.: Quiver representations respecting a quiver automorphism: a generalization of a theorem of Kac, *arXiv:math.RT/0203195*, 2002.
 - [Kac1] Kac, V.G.: Some remarks on nilpotent orbits, *J. of Algebra*, **64** (1980), 190-213.
 - [Kac2] Kac, V.G.: Infinite root systems, representations of graphs and Invariant Theory, *Inv. Math.* **56** (1980), 57-92.
 - [Kac3] Kac, V.G.: *Infinite dimensional Lie algebras* (3rd. ed), Cambridge University Press, Cambridge, 1990.
 - [Lit] Littelmann, P.: Koreguläre und äquidimensionale Darstellungen halbeinfacher Liegruppen, *J. of Algebra* **123** (1989), 1, 193-222.
 - [MWZ] Magyar, P., Weyman, J., and Zelevinsky, A.: Symplectic multiple flag varieties of finite type, *J. Algebra* **230**, (2000), 245-265.
- E-mail address:* mitia@mccme.ru